Jacobians

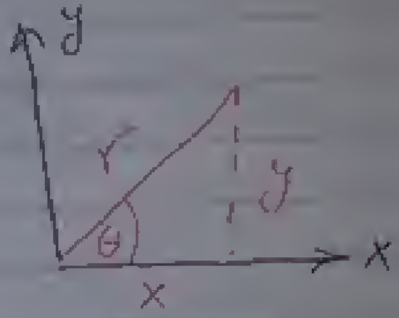
$x = r\cos\theta \to f(r,\theta)$ فاضل x بالنسبة r ثابت θ $y = r\sin\theta \to f(r,\theta)$

$$\rightarrow J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

المشتقة الأولى / x, y على بالنسبة لـ r, θ على

$$\rightarrow J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$



$x = r\cos\theta$
$y = r\sin\theta$

* implicit fn. دوال ضمنية

$F(u,v,x,y) = 0$ ، $G(u,v,x,y) = 0$

$$\Delta = J\left(\frac{F,G}{x,y}\right) =$$

$$\Delta_1 = J\left(\frac{F,G}{u,y}\right)$$

$$\Delta_2 = J\left(\frac{F,G}{x,u}\right)$$

$$\Rightarrow \frac{\partial f}{\partial [illegible]} = - \frac{\Delta_1}{\Delta}$$

$$\Rightarrow \frac{\partial G}{\partial u} = - \frac{\Delta_2}{\Delta}$$

$$\Rightarrow \frac{\partial F}{\partial u} = \frac{J\left(\frac{F,G}{u,y}\right)}{J\left(\frac{F,G}{x,y}\right)}$$

← المقام ثابت
والبسط زي ما هو بس
نغير x ونحط مكانها مقام ال u

$$\Rightarrow \frac{\partial F}{\partial v} = \frac{J\left(\frac{F,G}{x,y}\right)}{J\left(\frac{F,G}{x,y}\right)}$$

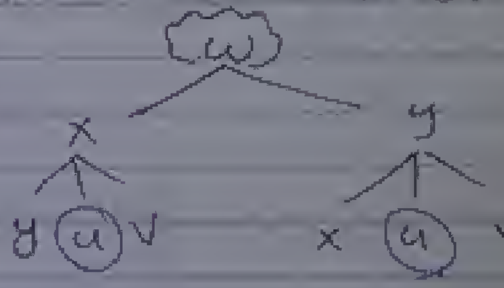
ex:- $\underset{(x,y)}{w} = x^2 y$, $u^2 - v = 3x + y$ } Coupled
$u - 2v^2 = x - 2y$

$F = u^2 - v - 3x - y = 0$

$G = u - x - 2v^2 + 2y = 0$

⇒ find $\frac{\partial w}{\partial u}$

Solution



$$\frac{\partial w}{\partial u} = \left(\frac{\partial w}{\partial x} * \frac{\partial x}{\partial u}\right) + \left(\frac{\partial w}{\partial y} * \frac{\partial y}{\partial u}\right)$$

$*\ \boxed{\frac{\partial w}{\partial x} = 2xy} \to ①$ → $\boxed{\frac{\partial w}{\partial y} = x^2} \to ②$

$$\rightarrow \frac{\partial x}{\partial u} = \ominus \frac{J\left(\frac{F,G}{u,y}\right)}{J\left(\frac{F,G}{x,y}\right)} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} -3 & -1 \\ -1 & 2 \end{vmatrix}} =$$

$$\therefore \frac{\partial x}{\partial u} = - \frac{4u+1}{-6-1} = \boxed{\frac{4u+1}{7}} \to ③$$

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متسبيش السالب

$$\frac{\partial y}{\partial u} = - \frac{J\left(\frac{F,G}{x,u}\right)}{J\left(\frac{F,G}{x,y}\right)}$$

$F(x,y,u,v)$
$G(x,y,u,v)$

* $w(x,u)$
التوابع - المتغير

$$\frac{\partial x}{\partial y} = \frac{\left(\frac{F,G}{y,u}\right)}{J\left(\frac{F,G}{x,y}\right)}$$

$$\therefore \frac{\partial y}{\partial u} = - \frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = - \frac{\begin{vmatrix} -3 & 2u \\ -1 & 1 \end{vmatrix}}{-7}$$

$$\therefore \frac{\partial y}{\partial u} = - \frac{-3+2u}{-7} = \boxed{\frac{2u-3}{7}} \quad (4)$$

$$\therefore \frac{\partial w}{\partial u} = ① * ③ + ② * ④$$

* Homogenous fn — الدالة المتجانسة

$f(x,y) = f(\lambda x, \lambda y) * \lambda^k$

ex ⇒ $f(x,y) = x^2 y \Rightarrow f(\lambda x, \lambda y) = \lambda^2 x^2 * \lambda y = \lambda^3 (x,y)$

then $f(x,y)$ is homogenous function of order k

Euler. theorem

If $f(x,y)$ is H.fn of order k

$\therefore x f_x + y f_y = k f$

$\therefore x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = k(k-1) f$

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Ex 35:

$$z = \sqrt{\frac{x^2 - xy + y^2}{yx^2 + y^2x}} \sin^{-1}\left(\frac{x+y}{x-y}\right)$$

* Prove that $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = \frac{3}{4} z$

{Solution}

$$z(x\lambda, y\lambda) = \sqrt{\frac{\lambda^2x^2 - \lambda^2xy + \lambda^2y^2}{\lambda^3yx^2 + \lambda^3xy^2}} \sin^{-1}\left(\frac{\lambda x + \lambda y}{\lambda(x-y)}\right)$$

$\therefore z(\lambda x, \lambda y) = \sqrt{\frac{1}{\lambda}} \; z = \lambda^{-1/2} z$

$\therefore f(z)$ is Homogenous of order $\frac{-1}{2}$

→ using Euler theorem.

$x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = \; k(k-1) f = \frac{-1}{2}\left(\frac{-1}{2} - 1\right) f$

$\therefore$ L.H.S $= \frac{-1}{2} \times \frac{-3}{2} z = \frac{3}{4} z =$ R.H.S

$x z_{xx} + y z_{xy} - 2 z_x = 0$

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Ex

$z = \frac{x^2 y^2}{x+y}$ prof that $x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{x \partial y} - 2 \frac{\partial z}{\partial x} = 0$

Solution

$$z(\lambda x, \lambda y) = \frac{\lambda^2 x^2 * \lambda^2 y^2}{\lambda x + \lambda y} = \frac{\lambda^4 x^2 y^2}{\lambda(x+y)} = \lambda^3 \frac{x^2 y^2}{x+y}$$

$\therefore$ z is an H. fn of order 3

using Eular theorem

$$x z_x + y z_y = k z = 3z \rightarrow (1)$$

diff eqn (1) w.r.t x

$$\therefore (x z_{xx} + z_x) + (y z_{yx}) = 3 z_x$$

$$\therefore x z_{xx} + y z_{xx} - 2 z_x = 0$$

C* Pg 46

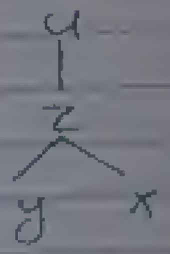
L.H.S = R.H.S

If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solution

Put $z = \frac{x^3+y^3}{x-y}$ $\qquad u = \tan^{-1} z$

$\qquad\qquad\qquad z = \tan u$



$\therefore z$ is an $\underline{\underline{H}}$ fn of order $\underline{\underline{2}}$

$\Longrightarrow x\, z_x + y\, z_y = 2z \longrightarrow ①$

$\Rightarrow L.H.S = x u_x + y u_y =$

$$= x\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + y\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial u}{\partial z}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$\therefore LHS = u_z\,(x z_x + y z_y)$ $\Rightarrow$ From ①

$\therefore L.H.S = u_z * 2z = \left(\frac{1}{1+z^2}\right) * 2z = \frac{1}{1+\tan^2 u} * 2\tan u$

$\therefore L.H.S = \frac{1}{\sec^2 u} * 2\tan u = \cos^2 u * 2\frac{\sin u}{\cos u} = 2\sin u\cos u$

$\therefore L.H.S = \sin 2u$ ※ $=$ R.H.S